§ Integral Calculus on Surfaces.

We have talked about differential calculus on surfaces.

Now, we move on to integration.

Question: Given a function $f: S \to \mathbb{R}$ on a surface S,

how to define $\int_S f$?

Some properties about integration on $\mathbb{R}^2$:

(1) For any bounded subset $\mathcal{U} \subseteq \mathbb{R}^2$,

$$\int_{\mathcal{U}} 1 = \text{Area}(\mathcal{U})$$

(2) Change of variable formula:

$$\boxed{\int_{\mathcal{U}} f(x,y)\, dx\, dy = \int_{\mathcal{U}'} f(u,v)\, |\text{Jac}\, \phi|\, du\, dv}$$

where $\phi: \mathcal{U}' \to \mathcal{U}$ is the change of coordinate transformation

$$\phi(u,v) = (x(u,v), y(u,v))$$

with Jacobian determinant

$$\text{Jac}\, \phi := \det(d\phi) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

E.g. $|\text{Jac}\, \phi| = r$ for polar coordinates r, θ

(3) $\exists$ various "Fundamental Theorems of Calculus"

Green's Theorem, Stokes' Theorem, Divergence Theorem

$$\boxed{\int_{\Omega} \text{"}d\text{"}\omega = \int_{\partial\Omega} \omega}$$

Def$^{\underline{\text{n}}}$: The support of a function $f : S \to \mathbb{R}$ is defined as

$$\operatorname{spt}(f) := \overline{\{ x \in S : f(x) \neq 0 \}} \quad \text{closure in } S$$

Def$^{\underline{\text{n}}}$: Let $\mathbf{X} : \mathcal{U} \xrightarrow{\cong} V \subseteq S$ be a parametrization of S

and $f : S \to \mathbb{R}$ be a function s.t. $\operatorname{spt}(f) \subseteq V$

Define

$$(\ast) \ \cdots \quad \int_S f := \int_{\mathcal{U}} f \circ \mathbf{X} \left\| \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} \right\| du\,dv$$

Remark: The def$^{\underline{\text{n}}}$ is independent of the choice of $\mathbf{X}$, i.e.

if $\mathbf{X}' : \mathcal{U}' \xrightarrow{\cong} V$ is another parametrization, then

$$\int_{\mathcal{U}} f \circ \mathbf{X} \left\| \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} \right\| du\,dv = \int_{\mathcal{U}'} f \circ \mathbf{X}' \left\| \frac{\partial \mathbf{X}'}{\partial u'} \times \frac{\partial \mathbf{X}'}{\partial v'} \right\| du'dv'$$

$\because$ change of variable formula (Ex: Prove this!)

<u>Note</u>: If the function f is not supported on a <u>single</u> coordinate neighborhood, i.e. $\mathrm{spt}(f) \not\subseteq V$, one can use a "partition of unity" to decompose into a (finite) sum

$$f = \sum_\alpha f_\alpha \qquad \text{s.t.} \quad \mathrm{spt}(f_\alpha) \subseteq V_\alpha$$

s.t. each f_α is contained in a single coord. nbd. V_α. Then,

$$\int_S f := \sum_\alpha \int_S f_\alpha$$

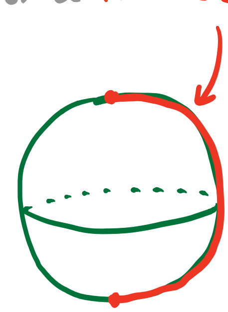
In practice, if $X : \mathcal{U} \to S$ is a parametrization which covers almost all of S (except a set of "measure zero") then $(*)$ is still applicable.

<u>Example</u>: (Area of the sphere)

The parametrization $X : (0, 2\pi) \times (0, \pi) \longrightarrow S^2(r)$

$$X(\theta, \varphi) := (r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r\cos\varphi)$$

covers almost the whole sphere $S^2(r)$ except for a latitude



$$\mathrm{Area}(S^2(r)) = \int_{S^2(r)} 1$$

$$= \int_0^\pi \int_0^{2\pi} 1 \cdot \underbrace{r^2 \sin\varphi}_{\left\| \frac{\partial X}{\partial \theta} \times \frac{\partial X}{\partial \varphi} \right\|} \, d\theta \, d\varphi$$

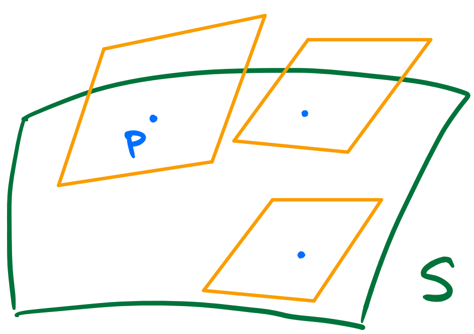
$$= 4\pi r^2.$$

§ First Fundamental Form

Recall that we have defined at each p on a surface S the tangent plane T_pS, which is a 2-dimensional subspace of $\mathbb{R}^3$. By putting all these tangent planes together, we get

tangent bundle of S : $\boxed{TS := \{(p, v) : p \in S,\ v \in T_pS\}}$

Note: We can think of TS as a 2-parameter family of 2-dimensional vector spaces (i.e. T_pS) parametrized by points p on S.



"disjoint union"

$$TS = \coprod_{p \in S} T_pS$$

Since each T_pS is a subspace of $\mathbb{R}^3$, it inherits the inner product from $\mathbb{R}^3$ as well. Therefore, we have the following:

Def$^{\underline{n}}$: The first fundamental form (1^{st} f.f.) of a surface at a point $p \in S$ is a positive definite, symmetric bilinear form (i.e. an inner product) defined on T_pS by

$$g_p : T_pS \times T_pS \longrightarrow \mathbb{R}$$

$$\boxed{g_p(u, v) := \langle u, v \rangle_{\mathbb{R}^3}}$$

standard inner product in $\mathbb{R}^3$

Note: TS is then a smooth family of inner product spaces parametrized by S.

We can express the 1^{st} f.f. locally as 2×2 matrices (g_{ij}) using coordinate systems as follows:

Given a parametrization $X(u_1, u_2) : \mathcal{U} \longrightarrow S$,

$$T_pS = \text{Span}\left\{ \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2} \right\}$$

we can express g_p by a matrix as

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad \text{where} \quad g_{ij} = \left\langle \frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_j} \right\rangle \quad (i,j = 1,2)$$

2×2 symmetric matrix

Therefore, if $u = a \frac{\partial}{\partial u_1} + b \frac{\partial}{\partial u_2}$

$v = c \frac{\partial}{\partial u_1} + d \frac{\partial}{\partial u_2}$ where $\frac{\partial}{\partial u_i} = \frac{\partial \mathbf{X}}{\partial u_i}$

then
$$g(u, v) = (a \quad b) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}.$$

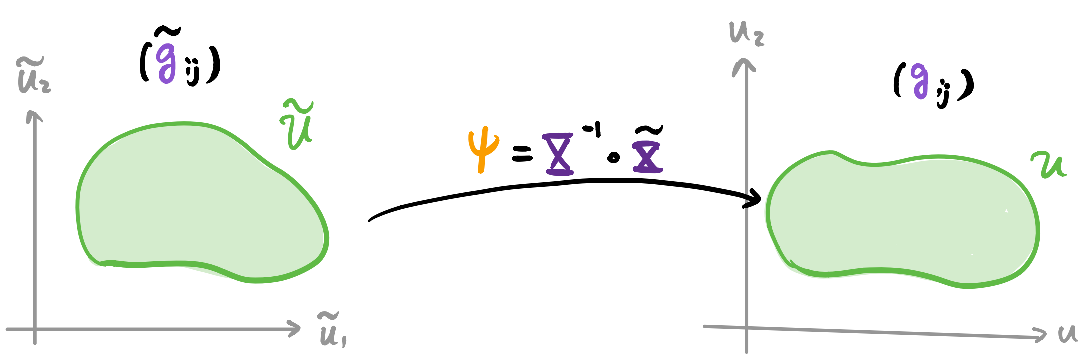
Question: How does the matrix (g_{ij}) transform when we change local coordinates?

Lemma: (Transformation law for (g_{ij}))

Suppose (g_{ij}) and $(\tilde{g}_{ij})$ are the 1st f.f. expressed in local coordinates $\mathbf{X}(u_1, u_2) : \mathcal{U} \to S$ and $\tilde{\mathbf{X}}(\tilde{u}_1, \tilde{u}_2) : \tilde{\mathcal{U}} \to S$ respectively. Then,

$$\boxed{(\tilde{g}_{ij}) = (D\psi)^T (g_{ij}) (D\psi)}$$

where $\psi = \mathbf{X}^{-1} \circ \tilde{\mathbf{X}} : \tilde{\mathcal{U}} \to \mathcal{U}$ is the transition map.



Proof: First of all,

$$(g_{ij}) = \begin{pmatrix} \left\langle \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_1} \right\rangle & \left\langle \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2} \right\rangle \\ \left\langle \frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_1} \right\rangle & \left\langle \frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_2} \right\rangle \end{pmatrix} \quad 2\times 2$$

$$= \underbrace{\begin{pmatrix} \text{---} & \frac{\partial X}{\partial u_1} & \text{---} \\ \text{---} & \frac{\partial X}{\partial u_2} & \text{---} \end{pmatrix}}_{2\times 3} \underbrace{\begin{pmatrix} | & | \\ \frac{\partial X}{\partial u_1} & \frac{\partial X}{\partial u_2} \\ | & | \end{pmatrix}}_{3\times 2} = (DX)^T (DX)$$

On the other hand,

$$(\tilde{g}_{ij}) = (D\tilde{X})^T (D\tilde{X})$$

$$= (D\psi)^T (DX)^T (DX)(D\psi) \qquad (\because \tilde{X} = X \circ \psi)$$

$$= (D\psi)^T (g_{ij}) (D\psi)$$

$\square$

Corollary: $\sqrt{\det(\tilde{g}_{ij})} = \sqrt{\det(g_{ij})}\ |\mathrm{Jac}\ \psi|$

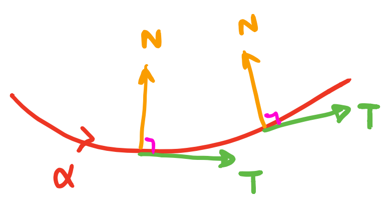
Note: $\sqrt{\det(g_{ij})} = \left\| \frac{\partial X}{\partial u_1} \times \frac{\partial X}{\partial u_2} \right\|$

$$\Rightarrow \int_S f \underset{\text{locally}}{=} \int_U f \underbrace{\sqrt{\det(g_{ij})}\, du_1 du_2}_{dA\,:\ \text{area form.}}$$

§ Gauss map & Second Fundamental Form

We now study the extrinsic geometry of surfaces and define various notions of curvatures for surfaces.

Recall: (Plane curves)

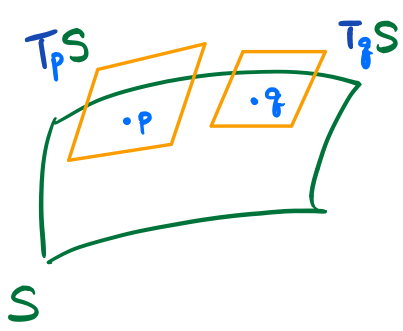


Frenet eq$^{\underline{n}}$: $\begin{pmatrix} T \\ N \end{pmatrix}' = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$

$\Downarrow$

$$\left[\begin{array}{ll} \text{Curvature } k & = \text{rate of change of } T \\ & = -(\text{rate of change of } N) \end{array} \right]$$

Now, for an (orientable) surface S, we consider its tangent bundle TS "=" a family of tangent planes T_pS



HOPE:

"Curvature of S" = rate of change of T_pS

= − rate of change of unit normal N_p

Note: In $\mathbb{R}^3$, a 2-dim'l subspace P is determined uniquely (up to a sign) by its unit normal $N \perp P$.

$\underline{\text{Def}^{\underline{n}}}$: Let $S \subseteq \mathbb{R}^3$ be an orientable surface, oriented by a global unit normal vector field called

$$N : S \longrightarrow S^2$$

unit sphere in $\mathbb{R}^3$

Gauss map

The Gauss map N is a smooth map from S to S^2

$\Rightarrow$ we can consider its differential at any $p \in S$

$$dN_p : T_pS \xrightarrow{\text{linear}} T_{N(p)}S^2 \cong N(p)^{\perp} = T_pS$$

$\underline{\text{Def}^{\underline{n}}}$: The shape operator / Weingarten map (at p) is the linear operator on T_pS defined by

$$\mathcal{S} = -dN_p : T_pS \xrightarrow{\text{linear}} T_pS$$

$\underline{\text{Def}^{\underline{n}}}$:

$$\boxed{H := \operatorname{tr} \mathcal{S}}$$ mean curvature

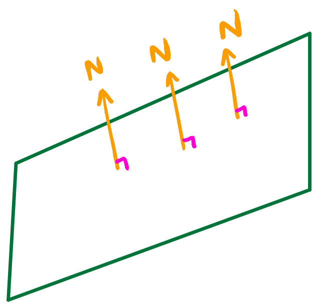
$$\boxed{K := \det \mathcal{S}}$$ Gauss curvature

Effect of orientation:

$$N \rightsquigarrow -N \Rightarrow \mathcal{S} \rightsquigarrow -\mathcal{S} \Rightarrow \underbrace{\begin{matrix} H \rightsquigarrow -H \\ K \rightsquigarrow K \end{matrix}}_{\text{unchanged !!}}$$

Examples:

(1) Planes

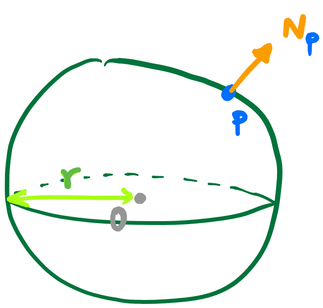


$N \equiv$ const. vector

$\Rightarrow S = -dN = 0$ i.e. $S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \boxed{\begin{matrix} H \equiv 0 \\ K \equiv 0 \end{matrix}}$ "flat"

(2) Spheres $S = \mathbb{S}^2(r) = \{ p \in \mathbb{R}^3 : \|p\| = r \}$



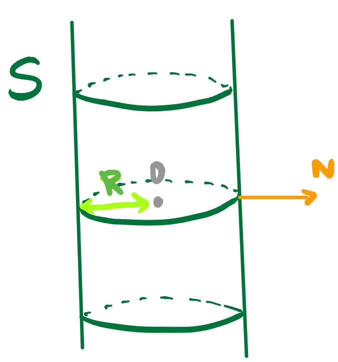
$S = \mathbb{S}^2(r)$

$N(p) = \dfrac{p}{\|p\|} = \dfrac{1}{r} p$

$\Rightarrow S = -dN = -\dfrac{1}{r} \mathrm{Id}$, i.e. $S = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}$

$\Rightarrow \boxed{\begin{matrix} H \equiv -\dfrac{2}{r} \\ K \equiv \dfrac{1}{r^2} \end{matrix}}$ Constant mean & Gauss curvature

(3) Cylinder $S = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = R^2 \}$



$N(x,y,z) = \dfrac{1}{R}(x,y,0)$

Question: How to compute

$S = -dN$?

Ans: Use local coordinates!

Locally, we can parametrize the cylinder by "cylindrical coordinates"

$$\mathbf{X}(\theta, z) := (R\cos\theta,\ R\sin\theta,\ z)$$

At each point on S, T_pS is spanned by

$$\begin{cases} \dfrac{\partial}{\partial\theta} := \dfrac{\partial \mathbf{X}}{\partial\theta} = (-R\sin\theta,\ R\cos\theta,\ 0) \\[2ex] \dfrac{\partial}{\partial z} := \dfrac{\partial \mathbf{X}}{\partial z} = (0, 0, 1) \end{cases}$$

Hence, a (local) unit normal vector field is

$$N(\theta, z) = \frac{\frac{\partial}{\partial\theta} \times \frac{\partial}{\partial z}}{\left\| \frac{\partial}{\partial\theta} \times \frac{\partial}{\partial z} \right\|} = (\cos\theta, \sin\theta, 0)$$

$$\Rightarrow \begin{cases} dN\left(\dfrac{\partial}{\partial\theta}\right) = \dfrac{\partial N}{\partial\theta} = (-\sin\theta, \cos\theta, 0) \\[2ex] dN\left(\dfrac{\partial}{\partial z}\right) = \dfrac{\partial N}{\partial z} = (0, 0, 0) \end{cases}$$

In terms of the basis $\left\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}\right\}$ for T_pS

$$S = -dN = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{H \equiv -\frac{1}{R}, \quad K \equiv 0}$$

Note: In all the examples above, S are represented by symmetric matrices. This is in fact a general phenomena.

Prop: $S = -dN_p : T_pS \to T_pS$ is a self adjoint operator on the inner product space $(T_pS, \langle , \rangle)$.

Proof: Take any parametrization $X(u,v)$ near p

$$T_pS = \text{span}\left\{ \frac{\partial X}{\partial u}, \frac{\partial X}{\partial v} \right\}$$

It suffices to prove

$$\boxed{\left\langle S\left(\frac{\partial X}{\partial u}\right), \frac{\partial X}{\partial v} \right\rangle = \left\langle \frac{\partial X}{\partial u}, S\left(\frac{\partial X}{\partial v}\right) \right\rangle} \quad (*)$$

By abuse of notation, we write

$$N(u,v) := N \circ X(u,v)$$

Since $N_p \perp T_pS$ <u>for all $p \in S$</u>,

$$\left\langle N, \frac{\partial X}{\partial v} \right\rangle \equiv 0$$

$$\xRightarrow[\text{w.r.t. } u]{\text{differentiate}} \left\langle \underbrace{\frac{\partial N}{\partial u}}_{dN\left(\frac{\partial X}{\partial u}\right) = -S\left(\frac{\partial X}{\partial u}\right)}, \frac{\partial X}{\partial v} \right\rangle + \left\langle N, \frac{\partial^2 X}{\partial u \partial v} \right\rangle \equiv 0$$

Hence, $\left\langle S\left(\frac{\partial X}{\partial u}\right), \frac{\partial X}{\partial v}\right\rangle = \left\langle N, \frac{\partial^2 X}{\partial u \partial v}\right\rangle$

Similarly, $\left\langle N, \frac{\partial X}{\partial u}\right\rangle \equiv 0$

$$\xRightarrow[\text{w.r.t. } v]{\text{differentiate}} \left\langle S\left(\frac{\partial X}{\partial v}\right), \frac{\partial X}{\partial u}\right\rangle = \left\langle N, \frac{\partial^2 X}{\partial v \partial u}\right\rangle$$

$$\boxed{\frac{\partial^2 X}{\partial u \partial v} = \frac{\partial^2 X}{\partial v \partial u}} \Longrightarrow (*)$$

"mixed partials are the same"

$\square$

<u>Def$^{\underline{n}}$</u>: The second fundamental form (at p) with respect to N is the symmetric bilinear form

$$A : T_pS \times T_pS \longrightarrow \mathbb{R}$$

$$\boxed{A(u,v) := \langle Su, v\rangle}$$

<u>Note</u>: In local coord., $T_pS = \text{span}\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right\}$

$$A = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{\text{symmetric matrix}} \quad \text{where } A_{ij} := A\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$$

By Spectral Theorem, the self adjoint operator

$$S = -dN_p : T_pS \longrightarrow T_pS \quad \text{diagonalizable}$$

eigenvalues : κ_1 , κ_2 principal curvatures

eigenvectors (unit) : v_1 , v_2 principal directions

(at p)

Def$^{\underline{n}}$: $p \in S$ is an umbilic point $\Leftrightarrow$ $\boxed{\kappa_1 = \kappa_2}$ at p

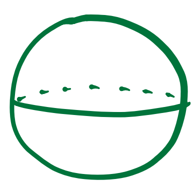
S is totally umbilic $\Leftrightarrow$ every $p \in S$ is umbilic.

Examples:

Plane

$\kappa_1 = \kappa_2 \equiv 0$

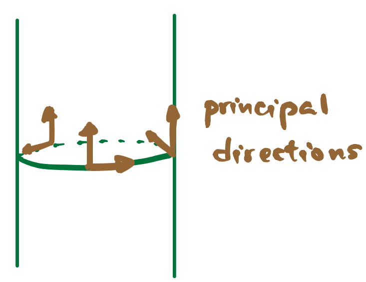
Sphere



$\kappa_1 = \kappa_2 \equiv -\frac{1}{R}$

totally umbilic

Cylinder



$\kappa_1 = -\frac{1}{R}$ $\quad \kappa_2 = 0$

not umbilic

§ Totally Umbilic Surfaces

Thm: $S \subseteq \mathbb{R}^3$ connected totally umbilic $\Longrightarrow$ S is contained in a plane or sphere.

Proof: Recall that totally umbilic means

$$\kappa_1(p) = \kappa_2(p) \quad \text{at every } p \in S$$

i.e. $\exists$ function $f : S \to \mathbb{R}$ s.t.

$$S = -dN_p = f(p)\,\mathrm{Id} : T_pS \to T_pS$$

Ex: Show that f is smooth!

For any parametrization $X(u,v)$ on S,

$$\begin{cases} S\left(\frac{\partial X}{\partial u}\right) = f \frac{\partial X}{\partial u} \\ S\left(\frac{\partial X}{\partial v}\right) = f \frac{\partial X}{\partial v} \end{cases} \Rightarrow \begin{cases} -\frac{\partial N}{\partial u} = f \frac{\partial X}{\partial u} \\ -\frac{\partial N}{\partial v} = f \frac{\partial X}{\partial v} \end{cases} \quad (*)$$

$$\Rightarrow \begin{cases} -\frac{\partial^2 N}{\partial v \partial u} = \frac{\partial f}{\partial v}\frac{\partial X}{\partial u} + f \frac{\partial^2 X}{\partial v \partial u} \\ -\frac{\partial^2 N}{\partial u \partial v} = \frac{\partial f}{\partial u}\frac{\partial X}{\partial v} + f \frac{\partial^2 X}{\partial u \partial v} \end{cases}$$

$$\Rightarrow \quad \frac{\partial f}{\partial v}\frac{\partial X}{\partial u} = \frac{\partial f}{\partial u}\frac{\partial X}{\partial v}$$

$$\Rightarrow \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial u} \equiv 0 \quad \left(\because \left\{\frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}\right\} \text{ lin. indep.}\right)$$

i.e. f is (locally) constant $(\because S \text{ connected})$

Case 1: $\boxed{f \equiv 0} \Rightarrow N \equiv \text{const.}$ plane!

Case 2: $\boxed{f \equiv c \neq 0}$

Claim: S is contained in a sphere of radius $\frac{1}{|c|}$

It suffices to show:

$$\mathbf{X} + \frac{1}{f} N \equiv \text{const. } p_0$$ ← center of the sphere

Note that:

$$\frac{\partial}{\partial u}\left(\mathbf{X} + \frac{1}{f} N\right) = \frac{\partial \mathbf{X}}{\partial u} + \frac{1}{f}\frac{\partial N}{\partial u} \overset{(*)}{\equiv} 0 .$$

Similarly,

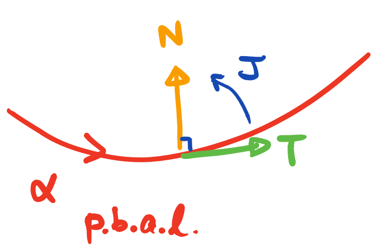
$$\frac{\partial}{\partial u}\left(\mathbf{X} + \frac{1}{f} N\right) = 0 .$$

This proves the claim since S is connected. $\square$

§ Normal curvatures

We now want to interpret the 2nd f.f. A as evaluating the curvature of certain plane curves lying on S.

Recall:



$$\boxed{k = \langle \alpha'', N \rangle}$$

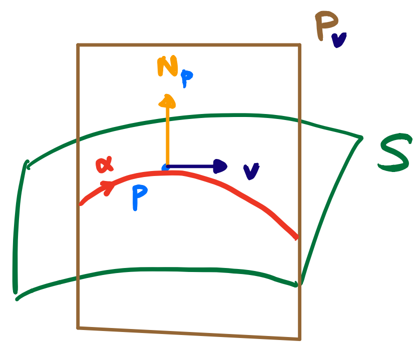
with orientation $\{T, N\}$

Let $S \subseteq \mathbb{R}^3$ be a surface oriented by N.

$\boxed{\text{Fix } p \in S \text{ and a unit tangent vector } v \in T_pS}$

Consider the oriented plane

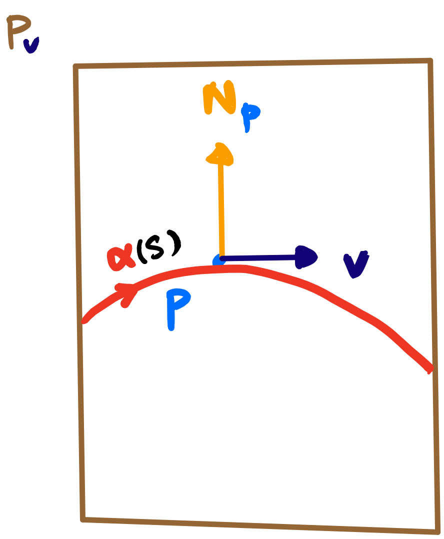
$$P_v = \text{span}\{\underbrace{v, N_p}_{\text{pos. orientation}}\}$$



which cuts S along some regular curve (why?) p.b.a.l.

$$\alpha : (-\varepsilon, \varepsilon) \longrightarrow S \quad \text{s.t.} \quad \alpha(0) = p, \quad \alpha'(0) = v$$

which can also be regarded as a plane curve on P_v



with curvature

$$\boxed{k_v = \langle \alpha''(0), N_p \rangle} \quad \text{— } (*)$$

On the other hand, since $\alpha \subseteq S$

$$\Rightarrow \quad \alpha'(s) \in T_{\alpha(s)}S, \quad \forall s$$

$$\Rightarrow \quad \langle \alpha'(s), N(\alpha(s)) \rangle \equiv 0 \quad \forall s$$

Differentiate w.r.t. s at $s=0$

$$\Longrightarrow \underbrace{\langle \alpha''(0), N_p \rangle}_{k_v} + \underbrace{\langle \alpha'(0), dN_p(\alpha'(0)) \rangle}_{-A(v,v)} = 0$$

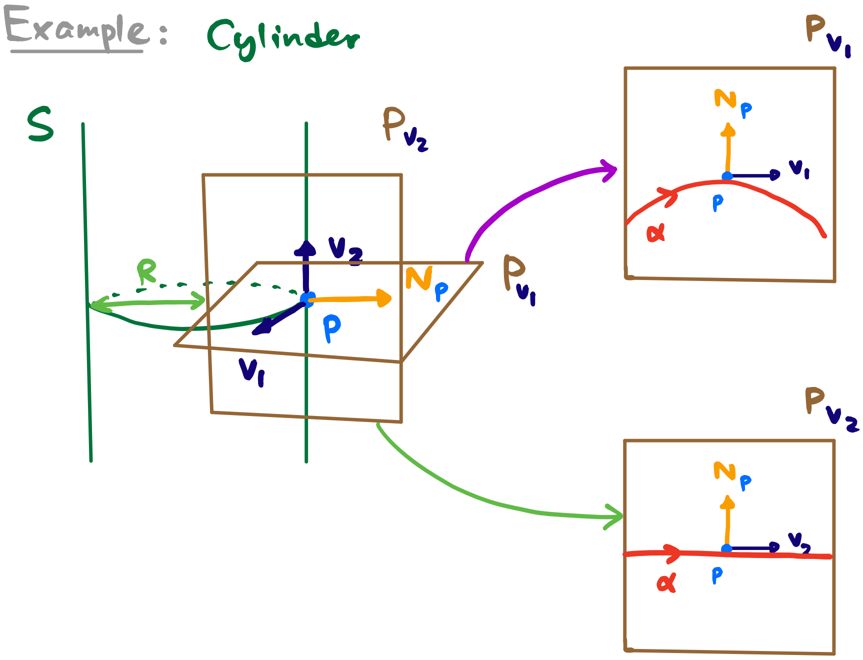
i.e. $A(v,v) = k_v$ (normal curvature along v)

By the variational characterization of eigenvalues,

the principal curvatures (at p) are

$$\boxed{\kappa_1 = \min_{\substack{v \in T_pS \\ \|v\|=1}} k_v} \quad \& \quad \boxed{\kappa_2 = \max_{\substack{v \in T_pS \\ \|v\|=1}} k_v}$$

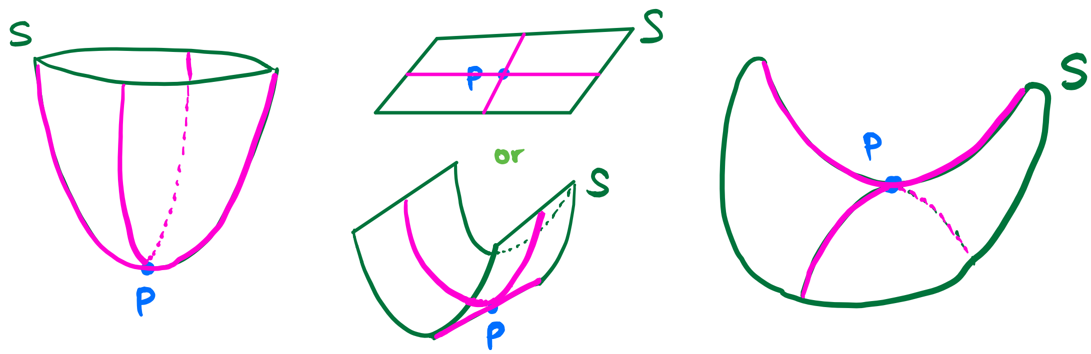
Example: Cylinder



$k_{v_1} = -\frac{1}{R}$

$k_{v_2} = 0$

We have the following local picture of surfaces:



$K > 0$	$K = 0$	$K < 0$
"elliptic"	"planar / parabolic"	"hyperbolic"

§ 2^{nd} fundamental form as Hessian

Let $f : S \longrightarrow \mathbb{R}$ be a smooth function.

Recall: $p \in S$ is a critical pt. of f

$$\Longleftrightarrow \quad 0 = df_p : T_pS \rightarrow \mathbb{R}$$

Question: Can one define 2^{nd} derivatives of f ?

Def$^{\underline{n}}$: The Hessian of f at a critical point $p \in S$ is defined as a map:

$$d^2 f_p : T_pS \longrightarrow \mathbb{R}$$

$$\boxed{d^2 f_p(v) := (f \circ \alpha)''(0)}$$

where $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$

$\alpha(0) = p$, $\alpha'(0) = v$

Lemma: $d^2 f_p$ is a well-defined quadratic form on T_pS.

Proof: Everything is "local". Fix any local coord. near p given by a parametrization $X(u,v)$, by abuse of notation,

$$f = f(u,v)$$

$$\alpha(t) = X(u(t), v(t))$$

$$v = \alpha'(0) = u'(0) \frac{\partial}{\partial u} + v'(0) \frac{\partial}{\partial v}$$

Hence, $(f \circ \alpha)(t) = f(u(t), v(t))$. Differentiate w.r.t. t,

$$(f \circ \alpha)'(t) = \frac{\partial f}{\partial u} \cdot u'(t) + \frac{\partial f}{\partial v} \cdot v'(t)$$

Differentiate again and evaluate at $t=0$,

$$(f \circ \alpha)''(0) = \left.\frac{\partial^2 f}{\partial u^2}\right|_p \cdot u'(0)^2 + 2 \left.\frac{\partial^2 f}{\partial u \partial v}\right|_p u'(0)\, v'(0) + \left.\frac{\partial^2 f}{\partial v^2}\right|_p \cdot v'(0)^2$$

$$+ \underbrace{\cancel{\left.\frac{\partial f}{\partial u}\right|_p}}_{0} \cdot u''(0) + \underbrace{\cancel{\left.\frac{\partial f}{\partial v}\right|_p}}_{0} \cdot v''(0)$$

($\because$ p is a crit. pt.)

Therefore, we have

$$d^2 f_p(v) = (f \circ \alpha)''(0) = \underbrace{\begin{pmatrix} u'(0) & v'(0) \end{pmatrix}}_{v^T} \underbrace{\left.\begin{pmatrix} \frac{\partial^2 f}{\partial u^2} & \frac{\partial^2 f}{\partial u \partial v} \\ \frac{\partial^2 f}{\partial v \partial u} & \frac{\partial^2 f}{\partial v^2} \end{pmatrix}\right|_p}_{\text{"Hess} f_p\text{"}} \underbrace{\begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}}_{v}$$

$\square$

Note: The Hessian is NOT well-defined as above if p is not a critical pt. There is a way to modify the definition so that it is well-defined at every point. That uses the 1st f.f. to define the concept of covariant derivative, which will be discussed later.

Since d^2f_p is just the usual Hessian of f in local coordinates, we also have 2nd derivative test for smooth funtions on surfaces.

Now, fix a point $p \in S$, we know that locally S is the graph of a function near p.

By translation & rotation, WLOG, we can assume $p = 0$ and near p

$$S = \{ z = f(x,y) \} \quad \text{where} \quad \begin{array}{l} f(0,0) = 0, \\ \nabla f(0,0) = 0. \end{array}$$

Hence, $T_pS = \{z = 0\} = \text{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\} = \mathbb{R}^2$.

Prop: $\boxed{A_p = \text{Hess} f(0,0)}$ —— $(*)$

Proof: Take any unit vector $v \in T_pS$, recall that

$$A_p(v,v) = k_v \leftarrow \text{normal curvature}$$

$$= \langle \alpha''(0), \underbrace{N(0)}_{=(0,0,1)} \rangle$$

$\alpha : (-\varepsilon, \varepsilon) \to S$
$\alpha(0) = 0$,
$\alpha'(0) = v$

$$= \text{Hess} f_{(0,0)}(v,v)$$

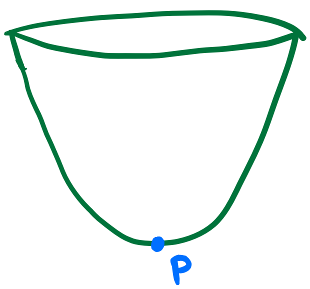
$\square$

Remark: (*) does **NOT** hold at points other than p !

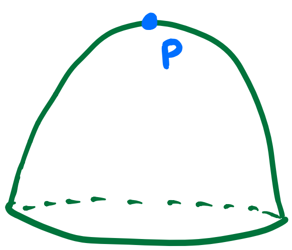
The proposition above implies the following local pictures:

$\boxed{K_p = \det A_p > 0}$

(i.e. A_p is pos./neg. definite)



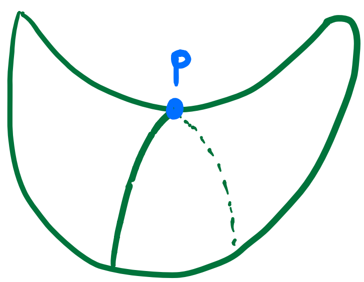
OR



"elliptic"

$\boxed{K_p = \det A_p < 0}$

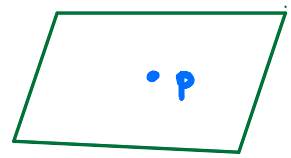
(i.e. A_p is indefinite)



"hyperbolic"

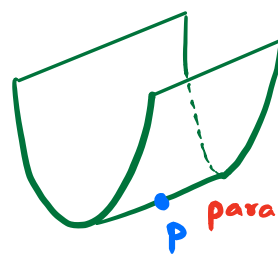
$\boxed{K_p = \det A_p = 0}$

(i.e. A_p is degenerate)



planar

OR



parabolic

OR

"Something else"